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Inequalities for the Associated Legendre Functions

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In this paper bounds for the associated Legendre functions of the first kind $P_n^m(x)$ for real $x \in [-1, 1]$ and integers m, n are proved. A relation is derived that allows us to generalize known bounds of the Legendre polynomials $P_n(x) \equiv P_n^0(x)$ for the Legendre functions $P_n^m(x)$ of non-zero order m. Furthermore, upper and lower bounds of the type $A(\alpha, n, m) \leqslant \max_{x \in [-1,1]} |(1-x^2)^{\alpha/2} P_n^m(x)| \leqslant B(\alpha, n, m)$ are proved for all $0 \leqslant \alpha \leqslant 1/2$ and $1 \leqslant |m| \leqslant n$. For $\alpha = 0$ and $\alpha = 1/2$ these upper bounds are improvements and simplifications of known results. © 1998 Academic Press

1. INTRODUCTION

The precise knowledge of upper limits on the associated Legendre functions [5, 9, 14] of the first kind $P_n^m(x)$ for real $x \in [-1, 1]$ and $n, m \in \mathbb{N}$ is essential to the investigation of many problems in theoretical physics, see, e.g., [11].

For the Legendre polynomials $P_n(x) \equiv P_n^0(x)$ several sharp estimations can be found in the literature. A classical result for $x \in (-1, 1)$ and $n \in \mathbb{N}$ is the improved version of Bernstein's inequality [1, 8]

$$|P_n(x)| < \sqrt{\frac{2}{\pi(n+1/2)}} \frac{1}{(1-x^2)^{1/4}}. \tag{1}$$

Inequalities for $P_n(x)$ that remain bounded for all $x \in [-1, 1]$ and $n \in \mathbb{N}$ have been derived by Martin [10]

$$|P_n(x)| \leqslant \frac{1}{[1+n(n+1)(1-x^2)]^{1/4}}, \tag{2}$$

and by Elbert and Laforgia [2]

$$|P_n(x)| \leqslant \frac{1}{[1+(\pi^4/16)(n+1/2)^4 (1-x^2)^2]^{1/8}}. \tag{3}$$

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For the Legendre functions $P_n^m(x)$ the upper limit which is usually cited in mathematical handbooks, e.g., [5, 9], is for $x \in (-1, 1)$ and $n, m \in \mathbb{N}$ with $n - m + 1 > 0$ given by

$$|P_n^{\pm m}(x)| < \sqrt{\frac{8}{\pi n}} \frac{(n \pm m)!}{n!} \frac{1}{(1 - x^2)^{m/2 + 1/4}}. \tag{4}$$

For fixed x, however, this limit turns out to be very crude for increasing m. This becomes evident, if (4) is compared with

$$\frac{2^{-1/2}}{\sqrt{n + 1/2}} \leqslant \max_{x \in [-1, 1]} |P_n^m(x)| \sqrt{\frac{(n - m)!}{(n + m)!}} \leqslant 2^{-1/2}, \tag{5}$$

which is valid for $1 \leqslant |m| \leqslant n$, and which shows that the dependence of $P_n^m(x)$ on n and m is essentially given by the square root over the factorials. The bounds in (5) can easily be derived from known relations of the Legendre functions. The left hand side follows immediately from the normalization integral (46) with use of the mean value theorem. The right hand side can, according to [4], be derived from the addition theorem (50) with $x' = x$, $\varphi = 0$, and under consideration that $P_n(1) = 1$. Another improved but less simple constant upper bound for $P_n^m(x)$ was proved in [7]

$$|P_n^m(x)| \sqrt{\frac{(n - m)!}{(n + m)!}} < \frac{\Gamma(1/4)\, e^{1/4}}{\pi} \left(\frac{1}{2n} + \frac{1}{2m} \right)^{1/(4 + 1/m)}. \tag{6}$$

A special type of x-dependent upper bound, which represents a generalization of Bernstein's inequality (1) to associated Legendre functions with $m \geqslant 1$, has been discussed by Szegö [15]. He conjectured that for fixed m and $n \to \infty$

$$|(1 - x^2)^{1/4}\, P_n^m(x)| \leqslant B(m) \cdot (n + 1/2)^{m - 1/2}, \tag{7}$$

where $B(m)$ shall depend on m only. This assumption was, however, already verified before by an inequality of Kogbetlianz [6, Eq. (23)] for ultraspherical polynomials, which has lately been rediscovered by Förster [3]. Transferred to Legendre functions with the use of (51) and under consideration of known Gamma function relations, this inequality reads

$$|(1 - x^2)^{1/4}\, P_n^m(x)| < \frac{2^{m+1}}{\sqrt{\pi}} \frac{\Gamma(n + 1/2)}{\Gamma(n - m + 1)}. \tag{8}$$

The conjectured behavior for $n \to \infty$ follows immediately from (53). Another result of this type for ultraspherical polynomials has been derived by Förster himself [3, Corollary 1.8]. Transferred to Legendre functions it reads

$$|(1-x^2)^{1/4}\, P_n^m(x)| < \frac{2^{m+1} m}{\sqrt{\pi}} \frac{\Gamma((n+m+1)/2)}{\Gamma((n-m)/2+1)}. \tag{9}$$

Finally, a further improved limit has recently been published by Martin [12]

$$|(1-x^2)^{1/4}\, P_n^m(x)| < \frac{\Gamma(n+1/2)}{\Gamma(n-m+1)}\, 2^{(m+1)^2/n} \sup_{0<t<\infty} |\sqrt{t}\, J_m(t)|, \tag{10}$$

the proof of which is, however, relatively intricate.

An x-dependent upper bound of the Legendre functions that remains finite for all $x \in [-1, 1]$ and $0 \leqslant |m| \leqslant n$ is, see, e.g., [7, Eq. 17],

$$|P_n^m(x)| \leqslant (1-x^2)^{m/2} \frac{2^{-m}}{m!} \frac{(n+m)!}{(n-m)!}. \tag{11}$$

Although reflecting the correct asymptotic behavior of $|P_n^m(x)|$ for $x \to \pm 1$, inequality (11) becomes, however, very poor near $x=0$. Provided that $n-m \geqslant 2$, this disadvantage is removed by [7, Eq. 12]

$$\begin{aligned} |P_n^m(x)| &\leqslant (1-x^2)^{m/2} \left[x^2 a_{n,m}^{2/(n-m)} + (1-x^2)\, b_{n,m}^{2/(n-m)}\right]^{(n-m)/2} \\ &\leqslant (1-x^2)^{m/2} \left[x^2 a_{n,m} + (1-x^2)\, b_{n,m}\right] \end{aligned} \tag{12}$$

with

$$a_{n,m} = \frac{2^{-m}}{m!} \frac{(n+m)!}{(n-m)!}, \qquad b_{n,m} = \frac{2^{-n} (n+m)!}{((n+m)/2)!\, ((n-m)/2)!}.$$

However, none of the last mentioned inequalities (8)–(12) has that basic n, m-dependence of (5).

This is different for the upper and lower limits of the Legendre functions derived in the following. In Theorem 2 we state for $1 \leqslant |m| \leqslant n$ and $0 \leqslant \alpha \leqslant 1/2$ general upper and lower limits of the type

$$A(\alpha, m) \leqslant \max_{x \in [-1,1]} |(1-x^2)^{\alpha/2}\, P_n^m(x)| \sqrt{\frac{(n-m)!}{(n+m)!}}\, (n+1/2)^{\alpha} \leqslant B(\alpha, m).$$

For $\alpha = 0$ and $\alpha = 1/2$ the right hand side can be compared with the constant upper bounds of (5) and (6) as well as with the bounds of Szegö's type (8)–(10). Furthermore, the sharpness of the upper limit $B(\alpha, m)$ can easily be checked on the lower limit $A(\alpha, m)$.

2. MAIN RESULTS

In the following, starting from the addition theorem (50), we prove at first a theorem that allows to transfer known upper bounds of the Legendre polynomials $P_n(x) \equiv P_n^0(x)$ to the Legendre functions $P_n^m(x)$ of non-zero order m.

THEOREM 1. *If for* $n \in \mathbb{N}$ *and real* $x \in [-1, 1]$ *the Legendre polynomials* $P_n(x)$ *have an upper limit of the form*

$$|P_n(x)| \leqslant u_n(x) := \frac{1}{[a_n + b_n(1-x^2)^k]^{1/(4k)}}, \tag{13}$$

where the constants $a_n, b_n \in \mathbb{R}$ *and* $k \in \mathbb{N}$ *satisfy* $a_n, b_n \geqslant 0$ *and* $a_n + b_n(1-x^2)^k > 0$, *then the associated Legendre functions* $P_n^m(x)$ *with* $m \in \mathbb{Z}$, $1 \leqslant |m| \leqslant n$ *are bounded by*

$$|P_n^m(x)| \leqslant \frac{2}{\sqrt{\pi}} \sqrt{\frac{(n+m)!}{(n-m)!}} \sqrt{u_n(x)}. \tag{14}$$

Inequality (14) has the basic n, m-dependence of (5). Note, that the relation between the factorials appearing in (14) can be simplified and clarified with help of the well known geometric-arithmetic mean inequality, see, e.g., [13],

$$\sqrt{\frac{(n+m)!}{(n-m)!}} \equiv \sqrt{\prod_{k=1}^{2m} (n-m+k)} < \left(\frac{1}{2m} \sum_{k=1}^{2m} (n-m+k)\right)^m = \left(n + \frac{1}{2}\right)^m. \tag{15}$$

Each of the three upper limits (1), (2), and (3) of the Legendre polynomials satisfies the conditions of Theorem 1 and can be transformed into inequalities for the associated Legendre functions via (14), as shown in the following corollaries.

COROLLARY 1. *For all real* $x \in (-1, 1)$ *and integer* n, m *with* $1 \leqslant |m| \leqslant n$ *the Legendre functions satisfy the inequality*

$$|P_n^m(x)| \leqslant \frac{2^{5/4}}{\pi^{3/4}} \sqrt{\frac{(n+m)!}{(n-m)!}} \frac{1}{(n+1/2)^{1/4}} \frac{1}{(1-x^2)^{1/8}}. \tag{16}$$

COROLLARY 2. *For all real $x \in [-1, 1]$ and integer n, m with $1 \leqslant |m| \leqslant n$ the Legendre functions satisfy the inequalities*

$$|P_n^m(x)| \leqslant \frac{2}{\sqrt{\pi}} \sqrt{\frac{(n+m)!}{(n-m)!}} \frac{1}{[1+n(n+1)(1-x^2)]^{1/8}} \tag{17}$$

and

$$|P_n^m(x)| \leqslant \frac{2}{\sqrt{\pi}} \sqrt{\frac{(n+m)!}{(n-m)!}} \frac{1}{[1+(\pi^4/16)(n+1/2)^4 (1-x^2)^2]^{1/16}}. \tag{18}$$

Theorem 1 is not only a simple tool to produce inequalities for higher order Legendre functions from those of the Legendre polynomials, but, as will be shown in the following theorem, the resulting inequalities, here especially (16), form also the basis for a refinement of several known bounds mentioned in the Introduction. Nevertheless, because of the square root over $u_n(x)$, inequality (14) is not yet optimal. This becomes evident when (52) is used to generate inequalities for the Bessel functions from those of the Legendre functions. With (47) and (53) we get for $x \geqslant 0$

$$|J_m(x)| = \lim_{n \to \infty} \left[n^m \frac{(n-m)!}{(n+m)!} \left| P_n^m \left(\cos \frac{x}{n} \right) \right| \right]$$

$$\leqslant \lim_{n \to \infty} \begin{cases} u_n \left(\cos \dfrac{x}{n} \right) & \text{for} \quad m = 0 \\ \dfrac{2}{\sqrt{\pi}} \sqrt{u_n \left(\cos \dfrac{x}{n} \right)} & \text{for} \quad m \geqslant 1. \end{cases}$$

Together with the bound of (3) this results for $m \geqslant 1$ in

$$|J_m(x)| \leqslant \frac{2/\sqrt{\pi}}{[1+(\pi^4/16)x^4]^{1/16}}, \tag{19}$$

and for $m = 0$ in

$$|J_0(x)| \leqslant \frac{1}{[1+(\pi^4/16)x^4]^{1/8}}. \tag{20}$$

Inequality (20), which has already previously been derived in [2], yields the perfect monotonic upper bound of $J_0(x)$ for $x \to \infty$, where $J_m(x) \sim \sqrt{2/(\pi x)} \cos(x - m\pi/2 - \pi/4)$. Compared to this (19) shows for $m \geqslant 1$ in the asymptotic case a weaker estimate only, essentially due to the smaller exponent in the denominator.

Together with a result of Martin [12], Corollary 1 creates the basis for the upper bound in the following theorem.

THEOREM 2. *For real* x *and* α *with* $x \in [-1, 1]$ *and* $0 \leqslant \alpha \leqslant 1/2$ *and integers* m, n *with* $1 \leqslant m \leqslant n$ *the function*

$$M^m_{n,\alpha}(x) := \max_{x \in [-1,1]} |(1-x^2)^{\alpha/2} P^m_n(x)| \sqrt{\frac{(n-m)!}{(n+m)!}} (n+1/2)^\alpha$$

satisfies

$$2^{-1/2}[1.11(m+1)]^{\alpha-1/2} < M^m_{n,\alpha}(x) < \frac{2^{5/4}}{\pi^{3/4}} \cdot \begin{cases} (m-\alpha)^{\alpha-1/4} & \text{if } \alpha \leqslant 1/4 \\ [1.11(m+1)]^{\alpha-1/4} & \text{if } \alpha \geqslant 1/4. \end{cases} \tag{21}$$

Via Eq. (47), this result can also be applied to Legendre functions of negative order m. In the case that m and n are not independent of each other and $n \to \infty$, it is sometimes more convenient to use the improved but less simple lower limit shown in (42) and (43). Whereas the upper and lower limits of (21) differ typically by a factor, $c \cdot m^{1/4}$ with constant c, the difference between the upper and the improved lower limit of (43) reduces to a constant in the special case that $m \to n$ for $n \to \infty$.

There are two interesting special cases of this theorem. For $\alpha = 0$ it results in:

COROLLARY 3. *For real* $x \in [-1, 1]$ *and integers* m, n *with* $1 \leqslant m \leqslant n$

$$\frac{1}{\sqrt{2.22(m+1)}} < \max_{x \in [-1,1]} |P^m_n(x)| \sqrt{\frac{(n-m)!}{(n+m)!}} < \frac{2^{5/4}}{\pi^{3/4}} \frac{1}{m^{1/4}}. \tag{22}$$

The right hand side of (22) improves and especially simplifies the constant upper bound given in (6). For $m \geqslant 5$ the corollary also improves the limits of (5). For $\alpha = 1/2$ the theorem results in:

COROLLARY 4. *For real* $x \in [-1, 1]$ *and integers* m, n *with* $1 \leqslant m \leqslant n$

$$\frac{1}{\sqrt{2}} < \max_{x \in [-1,1]} |(1-x^2)^{1/4} P^m_n(x)| \sqrt{\frac{(n-m)!}{(n+m)!}} \sqrt{n+1/2} < \frac{(1.11)^{1/4}\, 2^{5/4}}{\pi^{3/4}} (m+1)^{1/4}. \tag{23}$$

Under consideration of (15) it is evident that the right hand side satisfies Szegö's conjecture (7). Furthermore, numerical checks show that (23) provides generally a sharper upper limit than (8), (9), and (10) do.

3. PROOFS

3.1. *Proof of Theorem* 1

For integer degree n and integer order m the addition theorem (50) provides a simple tool for a transfer of known estimates of the Legendre polynomials $P_n(x) \equiv P_n^0(x)$ to the Legendre functions $P_n^m(x)$ of non-zero order m. Setting $x = x'$, Eq. (50) implies that for $0 \leqslant |m| \leqslant n$ and $x \in [-1, 1]$

$$[P_n^m(x)]^2 \frac{(n-m)!}{(n+m)!} = \frac{1}{\pi} \int_0^\pi P_n(x^2 + (1-x^2) \cos \varphi) \cos (m\varphi) \, d\varphi. \qquad (24)$$

This equation yields the basis for the proof of Theorem 1. Moreover it shows that $|P_n^m(x)|$ is an even function of x, so that proofs can be restricted to non-negative values of x. The following proof of Theorem 1 runs in refined form analogous to the one of [7, Theorem 2].

Proof. Under the conditions of Theorem 1

$$u_n(x^2 + (1-x^2) \cos \varphi) = \frac{1}{[a_n + b_n(1-x^2)^k \, (\sin^2 \varphi + x^2(1-\cos \varphi)^2)^k]^{1/(4k)}}$$

$$\leqslant \frac{1}{[a_n + b_n(1-x^2)^k \sin^{2k} \varphi]^{1/(4k)}} \leqslant \frac{u_n(x)}{\sqrt{\sin \varphi}}.$$

With (24) and (13), we get

$$[P_n^m(x)]^2 \frac{(n-m)!}{(n+m)!} \leqslant \frac{1}{\pi} \int_0^\pi u_n(x^2 + (1-x^2) \cos \varphi) \cdot |\cos(m\varphi)| \, d\varphi$$

$$\leqslant u_n(x) \cdot \frac{1}{\pi} \int_0^\pi \frac{|\cos(m\varphi)|}{\sqrt{\sin \varphi}} \, d\varphi.$$

Together with the result of the following lemma this verifies Theorem 1. ▮

LEMMA 1. *For all* $m \in \mathbb{N}$

$$I_m := \frac{1}{2} \int_0^\pi \frac{|\cos(m\varphi)|}{\sqrt{\sin \varphi}} \, d\varphi \leqslant 2.$$

Proof.

$$I_m = \int_0^{\pi/2} \frac{|\cos(m\varphi)|}{\sqrt{\sin\varphi}}\, d\varphi = \frac{1}{m}\int_0^{m\pi/2} \frac{|\cos\varphi|}{\sqrt{\sin(\varphi/m)}}\, d\varphi$$

$$= \frac{1}{m}\sum_{k=1}^{m} \int_0^{\pi/2} \frac{c_k(\varphi)}{\sqrt{\sin((\varphi + (k-1)\,\pi/2)/m)}}\, d\varphi, \tag{25}$$

where

$$c_k(\varphi) := \left|\cos\left(\varphi + (k-1)\frac{\pi}{2}\right)\right| = \begin{cases} \cos\varphi & \text{if} \quad k: \text{odd} \\ \sin\varphi & \text{if} \quad k: \text{even}. \end{cases}$$

(1) *Case* $m = 1$.

$$I_1 = \int_0^{\pi/2} \frac{\cos\varphi}{\sqrt{\sin\varphi}}\, d\varphi = 2\int_0^{\pi/2} \left(\frac{d}{d\varphi}\sqrt{\sin\varphi}\right) d\varphi = 2. \tag{26}$$

(2) *Case* $m \geqslant 2$. To prove this case, we need the following three different bounds.

(a) From (25) we get

$$I_m = \frac{1}{m}\int_0^{\pi/2} \frac{\cos\varphi}{\sqrt{\sin\varphi}} \sqrt{\frac{\sin\varphi}{\sin(\varphi/m)}}\, d\varphi$$

$$+ \frac{1}{m}\sum_{k=2}^{m} \int_0^{\pi/2} \frac{c_k(\varphi)}{\sqrt{\sin((\varphi + (k-1)\,\pi/2)/m)}}\, d\varphi.$$

Inequality (3.4.1) of [13] implies that $\sin\varphi/\sin(\varphi/m) \leqslant m$ for $0 \leqslant \varphi \leqslant \pi/2$. Together with (26) this leads to

$$I_m \leqslant \frac{2}{\sqrt{m}} + \frac{1}{m}\sum_{k=2}^{m} \int_0^{\pi/2} \frac{c_k(\varphi)}{\sqrt{\sin((\varphi + (k-1)\,\pi/2)/m)}}\, d\varphi. \tag{27}$$

(b) The monotonicity of the function in the denominator of the integrand allows an estimation of the integral in (27) by

$$I_m \leqslant \frac{2}{\sqrt{m}} + \frac{1}{m}\sum_{k=2}^{m} \frac{\int_0^{\pi/2} c_k(\varphi)\, d\varphi}{\sqrt{\sin((k-1)/m)(\pi/2))}}$$

$$= \frac{2}{\sqrt{m}} + \frac{1}{m}\sum_{k=1}^{m-1} \frac{1}{\sqrt{\sin((k/m)(\pi/2))}}. \tag{28}$$

(c) By the same reason the sum in (28) can be estimated by an integral

$$I_m \leqslant \frac{2}{\sqrt{m}} + \frac{1}{m}\int_0^m \frac{dx}{\sqrt{\sin((x/m)(\pi/2))}} - \frac{1}{m} = \frac{2}{\sqrt{m}} - \frac{1}{m} + \frac{2}{\pi}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{\sin\varphi}}$$
$$= \frac{2}{\sqrt{m}} - \frac{1}{m} + \frac{\Gamma(1/2)\,\Gamma(1/4)}{\pi\,\Gamma(3/4)} < \frac{2}{\sqrt{m}} - \frac{1}{m} + 1.67. \tag{29}$$

To prove the lemma for $m \geqslant 2$, the following three subcases are distinguished:

(a) *Case* $m = 2$. In this case Eq. (27) reads

$$I_2 \leqslant \sqrt{2} + \frac{1}{2}\int_0^{\pi/2} \frac{\sin\varphi\, d\varphi}{\sqrt{\sin((\varphi+\pi/2)/2)}} = \sqrt{2} + \frac{1}{2}\int_0^{\pi/2} \frac{\cos\varphi\, d\varphi}{\sqrt{\cos(\varphi/2)}}.$$

For $0 \leqslant \varphi \leqslant \pi/2$ the estimation $\cos(\varphi/2) = \sqrt{(1+\cos\varphi)/2} \geqslant \sqrt{\cos\varphi}$ yields

$$I_2 \leqslant \sqrt{2} + \frac{1}{2}\int_0^{\pi/2} \cos^{3/4}\varphi\, d\varphi = \sqrt{2} + \frac{\Gamma(1/2)\,\Gamma(7/8)}{4\Gamma(11/8)} < 1.96.$$

(b) *Case* $3 \leqslant m \leqslant 30$. Simple numerical calculations with sufficient precision of the finite sum on the right hand side of (28) show that for all m under consideration

$$I_m < 1.985.$$

(c) *Case* $m \geqslant 31$. To prove this final case inequality (29) is used. Since the right hand side of (29) is for $m \geqslant 1$ a monotonically decreasing function, we get for all m under consideration

$$I_m \leqslant \frac{2}{\sqrt{31}} - \frac{1}{31} + 1.67 < 1.997,$$

which finally completes the proof. ∎

3.2. *Proof of Theorem* 2

For fixed $n, m \in \mathbb{N}$ with $1 \leqslant m \leqslant n$ and $\alpha \in \mathbb{R}$ let

$$S_\alpha(x) := (1-x^2)^{\alpha/2}\, P_n^m(x). \tag{30}$$

Then, as can easily be checked with the help of (44) and (45), $S_\alpha(x)$ satisfies for $x \in (-1, 1)$ the differential equation

$$\frac{d}{dx}\left\{p_\alpha(x)\frac{d}{dx}S_\alpha(x)\right\} + q_\alpha(x)\, S_\alpha(x) = 0 \tag{31}$$

with

$$p_\alpha(x) := (1-x^2)^{1-\alpha}$$
$$q_\alpha(x) := \left[(n+1/2)^2 - (1/2-\alpha)^2 - \frac{m^2-\alpha^2}{1-x^2} \right] (1-x^2)^{-\alpha}. \tag{32}$$

LEMMA 2. *For $1 \leqslant m \leqslant n$ and $0 \leqslant \alpha \leqslant 1/2$:*

(1) *There exists an absolute maximum ξ_α of $|S_\alpha|$, defined by $|S_\alpha(\xi_\alpha)| = \max_{x \in [-1,1]} |S_\alpha(x)|$, which is contained in $[0, 1)$, and which corresponds to a local extremum of S_α.*

(2) *$q_\alpha(x) > 0$ for all $x \in [0, \xi_\alpha]$.*

(3) $$1 - \xi_\alpha^2 > \frac{m^2 - \alpha^2}{(n+1/2)^2 - (1/2-\alpha)^2} \geqslant \frac{(m-\alpha)^2}{(n+1/2)^2}. \tag{33}$$

(4) $$1 - \xi_\alpha^2 \leqslant 1 - \xi_{1/2}^2 < \frac{[1.11(m+1)]^2}{(n+1/2)^2}. \tag{34}$$

Proof. (1) Since $m \geqslant 1$, Eq. (24) implies that $S_\alpha(\pm 1) = P_n^m(\pm 1) = 0$ for all $0 \leqslant \alpha \leqslant 1/2$. Hence, due to the continuous differentiability, and since $P_n^m(x)$ with $m \leqslant n$ does not identically disappear for all $x \in [-1, 1]$, there are absolute maxima $\xi_{\alpha,i} \in (-1, 1)$, $i = 1, 2, ...,$ of $|S_\alpha(x)|$, defined by $|S_\alpha(\xi_\alpha)| = \max_{x \in [-1,1]} |S_\alpha(x)|$, with $S'_\alpha(\xi_{\alpha,i}) = 0$. Since according to (24) and (30), $|S_\alpha(x)|$ is an even function of x, there is a maximum $\xi_\alpha \in [0, 1)$. Due to $S_\alpha(\xi_\alpha) \neq 0$ and $q_\alpha(\xi_\alpha) > 0$, see point (2), Eq. (31) implies that $S''_\alpha(\xi_\alpha) \neq 0$, so that ξ_α corresponds to a local extremum of $S_\alpha(x)$.

(2) For this proof Eq. (31) is used. Partial integration results in

$$\int_{\xi_\alpha}^1 q_\alpha(x)\, S_\alpha^2(x)\, dx = -\int_{\xi_\alpha}^1 S_\alpha(x) \frac{d}{dx} \left\{ (1-x^2)^{1-\alpha} \frac{d}{dx} S_\alpha(x) \right\} dx$$
$$= \int_{\xi_\alpha}^1 (1-x^2)^{1-\alpha} \left[\frac{d}{dx} S_\alpha(x) \right]^2 dx > 0. \tag{35}$$

Suppose that $q_\alpha(\xi_\alpha) \leqslant 0$. Since $q_\alpha(0) > 0$ and $q_\alpha(x) \xrightarrow[x \to 1]{} -\infty$ and since the term in the brackets of (32) is a monotonically decreasing function for $x \in [0, 1]$, $q_\alpha(x)$ has exactly one zero within $(0, 1)$ and the above assumption implies that $q_\alpha(x) \leqslant 0$ for all $x \in [\xi_\alpha, 1)$. This, however, violates (35). Consequently, $q_\alpha(x) > 0$ for $x = \xi_\alpha$ and thus also for all $x \in [0, \xi_\alpha]$.

(3) The first inequality in (33) corresponds to the special case $q_\alpha(\xi_\alpha) > 0$. The second one is evident.

(4) For fixed α and variable τ with $0<\alpha, \tau<1/2$ differentiation of $S_\alpha^2(x)=(1-x^2)^{\alpha-\tau}\, S_\tau^2(x)$ with respect to x results for $x=\xi_\tau$ in

$$\frac{d}{dx} S_\alpha^2(x)\bigg|_{x=\xi_\tau} = -(\alpha-\tau)\, 2\xi_\tau(1-\xi_\tau^2)^{\alpha-\tau-1}\, S_\tau^2(\xi_\tau),$$

where $(d/dx)\, S_\tau^2(x)|_{x=\xi_\tau}=0$ has been utilized. Differentiation of this equation with respect to τ and transition $\tau\to\alpha$ yields

$$\frac{d^2}{dx^2} S_\alpha^2(x)\bigg|_{x=\xi_\alpha}\cdot\frac{d\xi_\tau}{d\tau}\bigg|_{\tau=\alpha} = 2\xi_\alpha(1-\xi_\alpha^2)^{-1}\, S_\alpha^2(\xi_\alpha)\geqslant 0.$$

Furthermore, since ξ_α corresponds to a local maximum of $S_\alpha^2(x)$, we have $(d^2/dx^2)\, S_\alpha^2(x)|_{x=\xi_\alpha}<0$. These results show that $d\xi_\alpha/d\alpha\leqslant 0$ for every $0<\alpha<1/2$, which proves the first inequality in (34).

The second inequality is an immediate consequence of a limit derived by Martin [12, Eqs. (16), (17)]. For $\cos\theta_{1/2} := \xi_{1/2}$, with $\theta_{1/2}\in[0, \pi/2]$, he proved that under the conditions of the present lemma

$$\theta_{1/2}<\frac{1.11(m+1)}{\sqrt{(n+1/2)^2-(m^2-1/4)(1/\sin^2\bar\theta-1/\bar\theta^2)}}, \tag{36}$$

where $\bar\theta$ can arbitrarily be chosen within $\theta_{1/2}\leqslant\bar\theta\leqslant\pi/2$. For $\bar\theta=\theta_{1/2}$ and due to $\theta_{1/2}>0$, inequality (36) is equivalent to

$$\begin{aligned}0&<\frac{(1.11(m+1))^2}{\theta_{1/2}^2}-(n+1/2)^2+(m^2-1/4)\left(\frac{1}{\sin^2\theta_{1/2}}-\frac{1}{\theta_{1/2}^2}\right)\\&<\frac{(1.11(m+1))^2}{\theta_{1/2}^2}-(n+1/2)^2+(1.11(m+1))^2\left(\frac{1}{\sin^2\theta_{1/2}}-\frac{1}{\theta_{1/2}^2}\right),\end{aligned}$$

which results in

$$\sin^2\theta_{1/2}<\left(\frac{1.11(m+1)}{n+1/2}\right)^2.$$

Inverse transform $\theta_{1/2}\to\xi_{1/2}$ finally confirms (34). ∎

3.2.1. *Proof of the Upper Limit.* Equation (16) of Corollary 1 leads to

$$\max_{x\in[-1,1]} |(1-x^2)^{\alpha/2}\, P_n^m(x)| = |S_\alpha(\xi_\alpha)| \leqslant \frac{2^{5/4}}{\pi^{3/4}}\sqrt{\frac{(n+m)!}{(n-m)!}}\,\frac{(1-\xi_\alpha^2)^{\alpha/2-1/8}}{(n+1/2)^{1/4}}.$$

Together with inequality (33) for $\alpha\leqslant 1/4$ and (34) for $\alpha\geqslant 1/4$ this result verifies the upper limit of Theorem 2.

3.2.2. *Proof of the Lower Limit.* First of all let us prove that for $1 \leqslant m \leqslant n$ the function

$$v^2(x) := S^2_{1/2}(x) + \frac{(p_{1/2}(x)\, S'_{1/2}(x))^2}{p_{1/2}(x)\, q_{1/2}(x)}, \tag{37}$$

which satisfies $v^2(\xi_{1/2}) = S^2_{1/2}(\xi_{1/2})$, is monotonically increasing for all $x \in [0, \xi_{1/2}]$. Since according to the above lemma $q_{1/2}(x) > 0$ and $p_{1/2}(x) > 0$ for these values of x, differentiation of (37) under consideration of (31)

$$(v^2(x))' = -(p_{1/2}(x)\, q_{1/2}(x))' \left(\frac{S'_{1/2}(x)}{q_{1/2}(x)}\right)^2 = 2x \frac{m^2 - 1/4}{(1-x^2)^2} \left(\frac{S'_{1/2}(x)}{q_{1/2}(x)}\right)^2 \geqslant 0$$

confirms this assumption.

Herewith, with the results of Lemma 2, and with Eq. (30) we get

$$\max_{x \in [-1, 1]} |(1-x^2)^{\alpha/2}\, P_n^m(x)| = |S_\alpha(\xi_\alpha)| \geqslant |S_\alpha(\xi_{1/2})| \equiv \frac{|S_{1/2}(\xi_{1/2})|}{(1-\xi^2_{1/2})^{1/4 - \alpha/2}}$$
$$= \frac{|v(\xi_{1/2})|}{(1-\xi^2_{1/2})^{1/4-\alpha/2}} \geqslant \frac{|v(0)|}{(1-\xi^2_{1/2})^{1/4-\alpha/2}}. \tag{38}$$

Under consideration of (37), (30), and of the special values of the Legendre functions in (48), and (49), $|v(0)|$ can be expressed by

$$|v(0)| = \begin{cases} w_{n,m} & \text{for} \quad n+m : \text{even} \\ \sqrt{\dfrac{(n+1/2)^2 - (m-1/2)^2}{(n+1/2)^2 - (m^2 - 1/4)}}\sqrt{(n-m+1)(n+m)} \cdot w_{n,m-1} & \\ & \text{for} \quad n+m: \text{odd}, \end{cases} \tag{39}$$

where for non-negative integers ν, μ with even values of $\nu - \mu$

$$w_{\nu,\mu} = \frac{(\nu+\mu)! \cdot 2^{-\nu}}{((\nu+\mu)/2)! \cdot ((\nu-\mu)/2)!}$$
$$\equiv \sqrt{\frac{(\nu+\mu)!}{(\nu-\mu)!}} \sqrt{\binom{\nu+\mu}{(\nu+\mu)/2} 4^{-(\nu+\mu)/2} \cdot \binom{\nu-\mu}{(\nu-\mu)/2} 4^{-(\nu-\mu)/2}}$$
$$\geqslant \sqrt{\frac{(\nu+\mu)!}{(\nu-\mu)!}} \frac{1}{\sqrt{2}} \cdot \begin{cases} (\nu^2 - \mu^2)^{-1/4} & \text{for} \quad \nu > \mu \geqslant 0 \\ \nu^{-1/4} & \text{for} \quad \nu = \mu \neq 0. \end{cases} \tag{40}$$

The lower estimate in (40) results from (54). Under consideration of $1 \leqslant m \leqslant n$ it becomes evident now that

$$|v(0)| \geqslant \begin{cases} w_{n,m} & \text{for } \quad n+m:\text{even} \\ \sqrt{(n-m+1)(n+m)} \cdot w_{n,m-1} & \text{for } \quad n+m:\text{odd} \end{cases}$$
$$\geqslant \sqrt{\frac{(n+m)!}{(n-m)!}} \frac{2^{-1/2}}{[n^2-(m-1)^2]^{1/4}}. \tag{41}$$

Herewith and with (34), the right hand side of (38) can be continued to

$$\max_{x \in [-1,1]} |(1-x^2)^{\alpha/2} P_n^m(x)| > \frac{1}{\sqrt{2}} \sqrt{\frac{(n+m)!}{(n-m)!}} \frac{[1.11(m+1)]^{\alpha-1/2}}{(n+1/2)^{\alpha}} \cdot c_{n,m} \tag{42}$$

with

$$c_{n,m} := \left[\frac{(n+1/2)^2}{n^2-(m-1)^2}\right]^{1/4} > \frac{1}{[1-((m-1)/n)^2]^{1/4}} > 1, \tag{43}$$

which just represents the lower limit of Theorem 2. ∎

In general, the second inequality in (43) represents only a harmless simplification of the expression for the lower limit. Under special circumstances, e.g., if $n \to \infty$ with $m/n \to 1$, this estimation becomes, however, very weak.

APPENDIX: BASIC FORMULAS

In the following a collection of basic formulas used in the main part of this work is listed.

The associated Legendre function of the first kind $P_n^m(x)$ is for real $x \in (-1, 1)$ and integer n, m with $0 \leqslant |m| \leqslant n$ the regular solution of the differential equation [5, 9]

$$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx} P_n^m(x)\right\} + r_{n,m}(x)\, P_n^m(x) = 0 \tag{44}$$

with

$$r_{n,m}(x) := n(n+1) - \frac{m^2}{1-x^2}, \tag{45}$$

which is normalized by

$$\frac{1}{2}\int_{-1}^{1} [P_n^m(x)]^2\,dx = \frac{1}{2n+1}\,\frac{(n+m)!}{(n-m)!}. \tag{46}$$

For $|m| > n$, the Legendre function is defined by $P_n^m(x) \equiv 0$.

Legendre functions of negative and positive integer order are interrelated by [5, 9]

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x). \tag{47}$$

For $x=0$ the Legendre functions can be expressed in terms of the factorials [5, Chap. 8.756]

$$P_n^m(0) = \cos\left(\frac{\pi}{2}(n+m)\right) \frac{(n+m)!\,2^{-n}}{((n+m)/2)!\,((n-m)/2)!}, \tag{48}$$

from which, via the known functional relations [5, Chap. 8.733], also the value of the derivative at this point can be derived

$$\left.\frac{d}{dx} P_n^m(x)\right|_{x=0} = (n-m+1)(n+m)\,P_n^{m-1}(0). \tag{49}$$

For $m=0$, the functions $P_n^0(x)$ correspond to the Legendre polynomials $P_n(x)$, which satisfy the addition theorem [5, 9]

$$\begin{aligned} &P_n(xx' + \sqrt{1-x^2}\,\sqrt{1-x'^2}\cos\varphi) \\ &\quad = P_n(x)\,P_n(x') + 2\sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} P_n^k(x)\,P_n^k(x')\cos(k\varphi). \end{aligned} \tag{50}$$

There is a close relation between the Legendre functions $P_n^m(x)$ and the ultraspherical polynomials $C_k^\lambda(x)$, see Refs. [5, 9]. For integers n, m with $0 \leqslant m \leqslant n$ it reads

$$P_n^m(x) = (-1)^m \frac{(2m)!}{m!\,2^m} (1-x^2)^{m/2}\,C_{n-m}^{m+1/2}(x). \tag{51}$$

Consequently, many results derived for the ultraspherical polynomials can immediately be transferred to the Legendre functions of integer degree and order.

Another relation, see Refs. [5, 9], exists between the Legendre functions and the Bessel functions for real $x \geqslant 0$ and integer $n, m \geqslant 0$

$$J_m(x) = \lim_{n \to \infty} \left\{ n^m P_n^{-m} \left(\cos \frac{x}{n} \right) \right\}.$$

For the Gamma function $\Gamma(x)$ and the factorials $n! = \Gamma(n+1)$ there exists the limit [5]

$$\lim_{n \to \infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1. \tag{53}$$

For $n \in \mathbb{N}$, a sharp lower bound for the binomial coefficients reads [13, Eq. 3.1.29]

$$\binom{2n}{n} \geqslant \frac{4^n}{\sqrt{4n}}. \tag{54}$$

REFERENCES

1. V. A. Antonov and K. V. Holševnikov, An estimate of the remainder in the expansion of the generating function for the Legendre polynomials (generalization and improvement of Bernstein's inequality), *Vestnik Leningrad. Univ. Mat.* **13** (1981), 163–166.
2. A. Elbert and A. Laforgia, An inequality for Legendre polynomials, *J. Math. Phys.* **35** (1994), 1348–1360.
3. K.-J. Förster, Inequalities for ultraspherical polynomials and application to quadrature, *J. Comput. Appl. Math.* **49** (1993), 59–70.
4. A. Fryant, Bounds of the Legendre functions, *Pure Appl. Math. Sci.* **23** (1986), 63–66.
5. I. S. Gradshteyn and I. M. Ryzhik, "Table of Integrals, Series, and Products," 5th ed., Academic Press, New York, 1994.
6. E. Kogbetlianz, Recherches sur la sommabilité des séries ultrasphériques par la méthode des moyennes arithméthiques, *J. Math. III* **2** (1924), 107–187.
7. G. Lohöfer, Inequalities for Legendre functions and Gegenbauer functions, *J. Approx. Theory* **64** (1991), 226–234.
8. L. Lorch, Alternative proof of a sharpened form of Bernstein's inequality for Legendre polynomials, *Appl. Anal.* **14** (1983), 237–240.
9. W. Magnus, F. Oberhettinger, and R. P. Soni, "Formulas and Theorems for the Special Functions of Mathematical Physics," Springer-Verlag, New York, 1966.
10. A. Martin, Unitarity and high-energy behavior of scattering amplitudes, *Phys. Rev.* **129** (1963), 1432–1436.
11. A. Martin, Elastic scattering and orthogonal polynomials, *in* "Orthogonal Polynomials and Their Applications" (C. Brezinski, L. Gori, and A. Ronveaux, Eds.), pp. 131–138, J. C. Baltzer AG, IMACS, 1991.

12. A. Martin, An optimal inequality on associated Legendre functions, *Ann. Numer. Math.* **2** (1995), 327–344.
13. D. S. Mitrinović, "Analytic Inequalities," Springer-Verlag, Berlin, 1970.
14. F. W. J. Olver, "Asymptotics and Special Functions," Academic Press, New York, 1974.
15. G. Szegö, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1975.